Lecture Notes on Integral Calculus

UBC Math 103 Lecture Notes by Yue-Xian Li (Spring, 2004)

1 Introduction and highlights

Differential calculus you learned in the past term was about differentiation. You may feel embarrassed to find out that you have already forgotten a number of things that you learned differential calculus. However, if you still remember that differential calculus was about *the rate of change, the slope of a graph, and the tangent of a curve*, you are probably OK.

- The essence of differentiation is *finding the ratio between the difference in the value of $f(x)$ and the increment in x.*

Remember, the derivative or the slope of a function is given by

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{1}$$

Integral calculus that we are beginning to learn now is called integral calculus. It will be mostly about *adding an incremental process to arrive at a "total"*. It will cover three major aspects of integral calculus:

1. The meaning of integration.
   - We'll learn that integration and differentiation are inverse operations of each other. They are simply two sides of the same coin (Fundamental Theorem of Caclulus).
2. The techniques for calculating integrals.
3. The applications.

2 Sigma Sum

2.1 Addition re-learned: adding a sequence of numbers

In essence, integration is an advanced form of addition. We all started learning how to add two numbers since as young as we could remember. You might say "Are you kidding? Are you telling me that I have to start my university life by learning addition?".

The answer is positive. You will find out that doing addition is often much harder than calculating an integral. Some may even find sigma sum is the most difficult thing to learn in integral calculus. Although this difficulty is by-passed by using the Fundamental Theorem of Caclulus, you should NEVER forget that *you are actually doing a sigma sum when you are calculating an integral.* This is one secret for correctly formulating the integral in many applied problems with ease!

Now, I use a couple of examples to show that your skills in doing addition still need improvement.

Example 1a: Find the total number of logs in a triangular pile of four layers (see figure).

Solution 1a: Let the total number be S_4, where 'S' stands for 'Sum' and the subscript reminds us that we are calculating the sum for a pile of 4 layers.

$$S_4 = \underbrace{1}_{\text{in layer 1}} + \underbrace{2}_{\text{in layer 2}} + \underbrace{3}_{\text{in layer 3}} + \underbrace{4}_{\text{in layer 4}} = 10.$$

A piece of cake!

Example 1b: Now, find the total number of logs in a triangular pile of 50 layers, i.e. find S_{50}! (Give me the answer in a few seconds without using a calculator).

Solution 1b: Let's start by formulating the problem correctly.

$$S_{50} = \underbrace{1}_{\text{in layer 1}} + \underbrace{2}_{\text{in layer 2}} + \cdots + \underbrace{49}_{\text{in layer 49}} + \underbrace{50}_{\text{in layer 50}} =?$$

where '$\cdots$' had to be used to represent the numbers between 3 and 48 inclusive. This is because there isn't enough space for writing all of them down. Even if there is enough space, it is tedious and unnecessary to write all of them down since the *regularity* of this sequence makes it very clear what are the numbers that are not written down.

Still a piece of cake? Not really if you had not learned Gauss's formula. We'll have to leave it unanswered at the moment.

Example 2: Finally, find the total number of logs in a triangular pile of k layers, i.e. find S_k (k is any positive integer, e.g. $k = 8,888,888$ is one possible choice)!

Solution 2: This is equivalent to calculating the sum of the first k positive integers.

$$S_k = 1 + 2 + \cdots + (k-1) + k.$$

The only thing we can say now is that the answer must be a function of k which is the total number of integers we need to add. Again, we have to leave it unanswered at the moment.

2.2 Regular vs irregular sequences

A **sequence** is a list of numbers written in a definite order. A sequence is *regular* if each term of the sequence is uniquely determined, following a well-defined rule, by its *position/order* in the sequence (often denoted by an integer i). Very often, each term can be generated by an explicit *formula* that is expressed as a function of the position i, e.g. $f(i)$. We can call this formula the *sequence generator* or the *general term*.

For example, the ith term in the sequence of integers is identical to its location in the sequence, thus its sequence generator is $f(i) = i$. Thus, the 9th term is 9 while the 109th term is equal to 109.

Example 3: The sum of the first ten odd numbers is

$$O_{10} = 1 + 3 + 5 + \cdots + 19.$$

Find the sequence generator.

Solution 3: Note that the ith odd number is equal to the ith even number minus 1. The ith even number is simply $2i$. Thus, the ith odd number is $2i - 1$, namely $f(i) = 2i - 1$. To verify, 5 is the 3rd odd number, i.e. $i = 3$. Thus, $2i - 1 = 2 \times 3 - 1 = 5$ which is exactly the number we expect.

Knowing the sequence generator, we can write down the sum of the first k odd numbers for any positive integer k.

$$O_k = 1 + 3 + 5 + \cdots + (2k - 1).$$

Example 4: Find the sequence generator of the following sum of 100 products of subsequent pairs of integers.

$$P_{100} = \underbrace{1 \cdot 2}_{\text{1st term}} + \underbrace{2 \cdot 3}_{\text{2nd term}} + \underbrace{3 \cdot 4}_{\text{3rd term}} + \cdots + \underbrace{100 \cdot 101}_{\text{100th term}}$$

Solution 4: Since the ith term is equal to the number i multiplied by the subsequent integer which is equal to $i + 1$. Thus, $f(i) = i(i + 1)$.

Knowing the sequence generator, we can write down the sum of k such terms for any positive integer k.

$$P_k = 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k + 1).$$

Example 4: The sequence of the first 8 digits of the irrational number $\pi = 3.1415926\ldots$ is

$$\Pi_8 = 3 + 1 + 4 + 1 + 5 + 9 + 2 + 6.$$

We cannot find the sequence generator since the sequence is irregular, we cannot express the sum of the first k digits of π (for arbitrary k).

2.3 The sigma notation

In order to short-hand the mathematical exression of the sum of a regular sequence, a convenient notation is introduced.

Definition (Σ sum): The sum of the first k terms of a sequence generated by the sequence generator $f(i)$ can be denoted by

$$S_k = f(1) + f(2) + \cdots + f(k) \equiv \sum_{i=1}^{k} f(i)$$

where the symbol Σ (the Greek equivalent of S reads "sigma") means **"take the sum of"**, the general expression for the terms to be added or the sequence generator $f(i)$ is called the **summand**, i is called the **summation index**, 1 and k are, respectively, the starting and the ending indices of the sum.

Thus,

$$\sum_{i=1}^{k} f(i)$$

means *calculate the sum of all the terms generated by the sequence generator $f(i)$ for all integers starting from $i = 1$ and ending at $i = k$.*

Note that the value of the sum is independent of the summation index i, hence i is called a "dummy" variable serving for the sole purpose of running the summation from the starting index to the ending index. Therefore, the sum only depends on the summand and both the starting and the ending indices.

Example 5: Express the sum $S_k = \sum_{i=3}^{k} i^2$ in an expanded form.

Solution 5: The sequence generator is $f(i) = i^2$. Note that the starting index is not 1 but 3!. Thus, the 1st term is $f(3) = 3^2$. The subsequent terms can be determined accordngly. Thus,

$$S_k = \sum_{i=3}^{k} i^2 = f(3) + f(4) + f(5) + \cdots + f(k) = 3^2 + 4^2 + 5^2 + \cdots + k^2.$$

An easy check for a mistake that often occurs. If you still find the "dummy" variable i in an expanded form or in the final evaluation of the sum, your answer must be WRONG.

2.4 Gauss's formula and other formulas for simple sums

Let us return to Examples 1 and 2 about the total number of logs in a triangular pile. Let's start with a pile of 4 layers. Imaging that you could (in a "thought-experiement") put an identical pile with up side down adjacent to the original pile, you obtain a pile that contains twice the number of logs that you want to calculate (see figure).

The advantage of doing this is that, in this double-sized pile, each layer contains an equal number of logs. This number is equal to number on the 1st (top) layer plus the number on the 4th (bottom) layer. In the mean time, the height of the pile remains unchanged (4). Thus, the number in this double-sized pile is $4 \times (4+1) = 20$. The sum S_4 is just half of this number which is 10.

Let's apply this idea to finding the formula in the case of k layers. Note that

$$
\begin{array}{rrclr}
\text{(Original)} & S_k & = & 1 \;+\; 2 \;+\cdots+\; k-1 \;+\; k. & (2) \\
\text{(Inversed)} & S_k & = & k \;+\; k-1 \;+\cdots+\; 2 \;+\; 1. & (3) \\
\text{(Adding the two)} & 2S_k & = & \underbrace{(k+1)+(k+1)+\cdots+(k+1)+(k+1)}_{\text{k terms in total}} = k(k+1). & (4)
\end{array}
$$

Dividing both sides by 2, we obtain Gauss's formula for the sum of the first k positive integers.

$$S_k = \sum_{i=1}^{k} i = 1 + 2 + \cdots + k = \frac{1}{2}k(k+1). \tag{5}$$

This actaully answered the problem in Example 2. The answer to Example 1b is even simpler.

$$S_{50} = \sum_{i=1}^{50} i = \frac{1}{2} \times 50 \times 51 = 1275.$$

The following are two important simple sums that we shall use later. One is the sum of the first k integers squared.

$$S_k = \sum_{i=1}^{k} i^2 = 1^2 + 2^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1). \tag{6}$$

The other is the sum of the first k integers cubed.

$$S_k = \sum_{i=1}^{k} i^3 = 1^3 + 2^3 + \cdots + k^3 = \left[\frac{1}{2}k(k+1)\right]^2. \tag{7}$$

We shall not illustrate how to derive these formulas. You can find it in numerous calculus text books.

To prove that these formulas work for arbitrarily large integers k, we can use a method called mathematical induction. To save time, we'll just outline the basic ideas here.

The only way we can prove things concerning arbitrarily large numbers is to guarantee that this formula must be correct for $k = N + 1$ if it is correct for $k = N$. This is like trying to arrange a string/sequence of standing dominos. To guarantee that all the dominos in the string/sequence fall one after the other, we need to guarantee that the falling of each domino will necessarily cause the falling of the subsequent one. This is the essence of proving that if the formula is right for $k = N$, it must be right for $k = N + 1$. The last thing you need to do is to knock down the first one and keep your fingers crossed. Knocking down the first domino is equivalent to proving that the formula is correct for $k = 1$ which is very easy to check in all cases (see Keshet's notes for a prove of the sum of the first k integers squared).

2.5 Important rules of sigma sums

Rule 1: Summation involving constant summand.

$$\sum_{i=1}^{k} c = \underbrace{c + c + \cdots + c}_{\text{k terms in total}} = kc.$$

Note that: the total # of terms=ending index - starting index +1.

Rule 2: Constant multiplication: multiplying a sum by a constant is equal to multiplying each term of the sum by the same constant.

$$c\sum_{i=1}^{k} f(i) = \sum_{i=1}^{k} cf(i)$$

Rule 3: Adding two sums with identical starting and ending indices is equal to the sum of sums of the corresponding terms.

$$\sum_{i=1}^{k} f(i) + \sum_{i=1}^{k} g(i) = \sum_{i=1}^{k}[f(i) + g(i)].$$

Rule 4: Break one sum into more than one pieces.

$$\sum_{i=1}^{k} f(i) = \sum_{i=1}^{n} f(i) + \sum_{i=n+1}^{k} f(i), \quad (1 \le n < k).$$

Example 7: Let's go back to solve the sum of the first k odd numbers.

Solution 7:

$$O_k = 1 + 3 + \cdots + (2k - 1) = \sum_{1}^{k}(2i - 1) = \sum_{1}^{k}(2i) - \sum_{1}^{k} 1 = 2\sum_{1}^{k} i - k,$$

where Rules 1, 2, and 3 are used in the last two steps. Using the known formulas, we obtain

$$O_k = \sum_1^k (2i-1) = k(k+1) - k = k^2.$$

Example 8: Let us now go back to solve Example 4. It is the sum of the first k products of pairs of subsequent integers.

Solution 8:

$$P_k = 1 \cdot 2 + 2 \cdot 3 + \cdots + k \cdot (k+1) = \sum_1^k i(i+1) = \sum_1^k [i^2 + i] = \sum_1^k i^2 + \sum_1^k i,$$

where Rule 3 was used in the last step. Applying the formulas, we learned

$$P_k = \sum_1^k i(i+1) = \frac{1}{6}k(k+1)(2k+1) + \frac{1}{2}k(k+1) = \frac{1}{3}k(k+1)(k+2).$$

This is another simple sum that we can easily remember.

Example 9: Calculate the sum $S = \sum\limits_1^k (i+2)^3$.

Solution 9: This problem can be solved in two different ways. The first is to expand the summand $f(i) = (i+2)^3$ which yield

$$S = \sum_1^k (i+2)^3 = \sum_1^k (i^3 + 6i^2 + 12i + 8) = \sum_1^k i^3 + 6\sum_1^k i^2 + 12\sum_1^k i + 8k.$$

We can solve the resulting three sums separately using the known formulas.

But there is a better way to solve this. This involves substituting the summation index. We find it easier to see how substitution works by expanding the sum.

$$S = \sum_{i=1}^k (i+2)^3 = 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 + (k+2)^3.$$

We see that this is simply a sum of integers cubed. But the sum does not start at 1^3 and end at k^3 like in the formula for the sum of the first k integers cubed.

Thus we re-write the sum with a sigma notation with an new index called l which starts at $l = 3$ and ends at $l = k + 2$ (there is no need to change the symbol for the index, you can keep calling it i if you do not feel any confusion). Thus,

$$S = \sum_{i=1}^k (i+2)^3 = 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 + (k+2)^3 = \sum_{l=3}^{k+2} l^3.$$

We just finished doing a substitution of the summation index. It is equivalent to replacing i by $l = i + 2$. This relation also implies that $i = 1 \;\Rightarrow\; l = 3$ and $i = k \;\Rightarrow\; l = k + 2$. This is actually how you can determine the starting and the ending values of the new index.

Now we can solve this sum using Rule 4 and the known formula.

$$S = \sum_{i=1}^{k}(i+2)^3 = \sum_{l=3}^{k+2} l^3 = \sum_{l=1}^{k+2} l^3 - \sum_{l=1}^{2} l^3 = \left[\frac{1}{2}(k+2)(k+3)\right]^2 - 1^2 - 2^3 = \left[\frac{1}{2}(k+2)(k+3)\right]^2 - 9.$$

2.6 Applications of sigma sum

The area under a curve

We know that the area of a rectangle with length l and width w is $A_{rect} = w \cdot l$.

Starting from this formula we can calculate the area of a triangle and a trapezoid. This is because a triangle and a trapezoid can be transformed into a rectangle (see Figure). Thus, for a triangle of height h and base length b

$$A_{trig} = \frac{1}{2}hb.$$

Similarly, for a trapezoid with base length b, top length t, and height h

$$A_{trap} = \frac{1}{2}h(t+b).$$

Following a very similar idea, the sum of a trapezoid-shaped pile of logs with t logs on top layer, b logs on the bottom layer, and a height of $h = b - t + 1$ layers (see figure) is

$$\sum_{i=t}^{b} i = t + (t+1) + \cdots + (b-1) + b = \frac{1}{2}h(t+b) = \frac{1}{2}(b-t+1)(t+b). \tag{8}$$

Now returning to the problem of calculating the area. Another important formula is for the area of a circle of radius r.

$$A_{circ} = \pi r^2.$$

Now, once we learned sigma and/or integration, we can calculate the area under the curve of any function that is integrable.

Example 10: Calculate the area under the curve $y = x^2$ between 0 and 2 (see figure).

Solution 10: Remember always try to reduce a problem that you do not know how to solve into a problem that you know how to solve.

Let the area be A. Let's divide A into 3 rectangles of equal width $w = 2/3$. Thus,

$$A \approx w \cdot h_1 + w \cdot h_2 + w \cdot h_3 = \begin{cases} w[x_0^2 + x_1^2 + x_2^2] = \frac{1}{3}[0^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2], & \text{left} - \text{end approx.;} \\ w[x_1^2 + x_2^2 + x_3^2] = \frac{1}{3}[\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{3}{3}\right)^2], & \text{right} - \text{end approx..} \end{cases}$$

By using these rectangles, we introduced large errors in our estimates using the heights based on both the left and the right endpoints of the subintervals. To increase accurary, we need to increase the number of rectangles by making each one thinner. Let us now divide A into n rectangles of equal width $w = 2/n$. Thus, using the height based on the right endpoint of each subinterval, we obtain

$$A \approx S_n = A_1 + A_2 + \cdots + A_n = \sum_{i=1}^{n} w \cdot h_i = w \sum_{i=1}^{n} f(x_i) = \frac{2}{n} \sum_{i=1}^{n} x_i^2.$$

It is very important to keep a clear account of the height of each rectangle. In this case, $h_i = f(x_i) = x_i^2$. Thus, the key is finding the x-coordinate of the right-end of each rectangle. For rectangles of of equal width, $x_i = x_0 + iw = x_0 + i\Delta x/n$, where Δx is the length of the interval x_0 is the left endpoint of the interval. $x_0 = 0$ and $\Delta x = 2$ for this example. Therefore, $x_i = i(2/n)$. Substitute into the above equation

$$A \approx S_n = \frac{2}{n} \sum_{i=1}^{n} x_i^2. = \frac{2}{n} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 = \frac{8}{n^3} \sum_{i=1}^{n} i^2 = \frac{8}{n^3}\frac{1}{6} n(n+1)(2n+1) = \frac{4}{3}\frac{(n+1)(2n+1)}{n^2}.$$

Note that if we divide the area into infinitely many retangles with a width that is infinitely small, the approximate becomes accurate.

$$A = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{4}{3}\frac{(n+1)(2n+1)}{n^2} = \frac{8}{3}.$$

The volume of a solid revolution of a curve

We know the volume of a rectangular block of height h, width w, and length l is

$$V_{rect} = w \cdot l \cdot h.$$

The volume of a cylinder of thichness h and radius r

$$V_{cyl} = A_{cross} \cdot h = \pi r^2 h.$$

Example 11: Calculate the volume of the bowl-shaped solid obtained by rotating the curve $y = x^2$ on $[0, 2]$ about the y-axis.

2.7 The sum of a geometric sequence

3 The Definite Integrals and the Fundamental Theorem

3.1 Riemann sums

Definition 1: Suppose $f(x)$ is finite-valued and piecewise continuous on $[a, b]$. Let $P = \{x_0 = a, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a, b]$ into n subintervals $I_i = [x_{i-1}, x_i]$ of width $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, n$. (Note in a special case, we can partition it into subintervals of equal width: $\Delta x_i = x_i - x_{i-1} = w = (b-a)/n$ for all i). Let x_i^* be a point in I_i such that $x_{i-1} \leq x_i^* \leq x_i$. (Here are some special ways to choose x_i^*: (i) left endpoint rule $x_i^* = x_{i-1}$, and (ii) the right endpoint rule $x_i^* = x_i$).

The Riemann sums of $f(x)$ on the interval $[a, b]$ are defined by:

$$R_n = \sum_{i=1}^{n} h_i \Delta x_i = \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

which approximate the area between $f(x)$ and the x-axis by the sum of the areas of n thin rectangles (see figure).

Example 1: Approximate the area under the curve of $y = e^x$ on $[0, 1]$ by 10 rectangles of equal width using the left endpoint rule.

Solution 1: We are actually calculating the Riemann sum R_{10}. The width of each rectangle is $\Delta x_i = w = 1/10 = 0.1$ (for all i). The left endpoint of each subinterval is $x_i^* = x_i - 1 = (i-1)/10$. Thus,

$$R_{10} = \sum_{i=1}^{10} f(x_i^*) \Delta x_i = \sum_{i=1}^{10} e^{(i-1)/10}(0.1) = 0.1 \sum_{j=0}^{9} e^{j/10} = 0.1 \frac{1-e}{1-e^{1/10}} \approx 1.6338.$$

3.2 The definite integral

Definition 1: Suppose $f(x)$ is finite-valued and piecewise continuous on $[a, b]$. Let $P = \{x_0 = a, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a, b]$ with a length defined by $|p| = Max_{1 \leq i \leq n}\{\Delta x_i\}$ (i.e. the longest of all subintervals). The definite integral of $f(x)$ on $[a, b]$ is

$$\int_a^b f(x)dx = \lim_{|p| \to 0;\ n \to \infty} R_n = \lim_{|p| \to 0;\ n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

where the symbol $\int$ means **"to integrate"**, the function $f(x)$ to be integrated is called the **integrand**, x is called the **integration variable** which is a "dummy" variable, a and b are, respectively, the lower (or the starting) limit and the upper (or the ending) limit of the integral. Thus,

$$\int_a^b f(x)dx$$

means *integrate the function $f(x)$ starting from $x = a$ and ending at $x = b$.*

Example 2: Calculate the definite integral of $f(x) = x^2$ on [0.2]. (This is Example 10 of Lecture 1 reformulated in the form of a definite integral).

Solution 2:

$$I = \int_0^2 x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(\frac{2i}{n}\right)^2}_{f(x_i^*)} \underbrace{\left(\frac{2}{n}\right)}_{\Delta x_i} = \lim_{n\to\infty} \left(\frac{8}{n^3}\right) \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{4}{3} \frac{(n+1)(2n+1)}{n^2} = \frac{8}{3}.$$

Important remarks on the relation between an area and a definite integral:

- An area, defined as the physical measure of the size of a 2D domain, is always non-negative.
- The value of a definite integral, sometimes also referred to as an "area", can be both positive and negative.
- This is because: a definite integral = the limit of Riemann sums. But Riemann sums are defined as

$$R_n = \sum_{i=1}^{n} (\text{area of i}^{\text{th}} \text{ rectangle}) = \sum_{i=1}^{n} \underbrace{f(x_i^*)}_{height} \underbrace{\Delta x_i}_{width}.$$

- Note that both the hieght $f(x_i^*)$ and the width Δx_i can be negative implying that R_n can have either signs.

3.3 The fundamental theorem of calculus

3.4 Areas between two curves

4 Applications of Definite Integrals: I

4.1 Displacement, velocity, and acceleration

4.2 Density and total mass

4.3 Rates of change and total change

4.4 The average value of a function

5 Differentials

Definition: The *differential*, dF, of any differentiable function F is an infinitely small increment or change in the value of F.

Remark: dF is measured in the same units as F itself.

Example: If x is the position of a moving body measured in units of m (meters), then its *differential*, dx, is also in units of m. dx is an infinitely small increment/change in the position x.

Example: If t is time measured in units of s (seconds), then its *differential*, dt, is also in units of s. dt is an infinitely small increment/change in time t.

Example: If A is area measured in units of m^2 (square meters), then its *differential*, dA, also is in units of m^2. dA is an infinitely small increment/change in area A.

Example: If V is volume measured in units of m^3 (cubic meters), then its *differential*, dV, also is in units of m^3. dV is an infinitely small increment/change in volume V.

Example: If v is velocity measured in units of m/s (meters per second), then its *differential*, dv, also is in units of m/s. dv is an infinitely small increment/change in velocity v.

Example: If C is the concentration of a biomolecule in our body fluid measured in units of M ($1\ M = 1\ molar = 1\ mole/litre$, where 1 mole is about 6.023×10^{23} molecules), then its *differential*, dC, also is in units of M. dC is an infinitely small increment/change in the concentration C.

Example: If m is the mass of a rocket measured in units of kg (kilograms), then its *differential*, dm, also is in units of kg. dm is an infinitely small increment/change in the mass m.

Example: If $F(h)$ is the culmulative probability of finding a man in Canada whose height is smaller than h ($meters$), then dF is an infinitely small increment in the probability.

Definition: The derivative of a function F with respect to another function x is defined as the quotient between their differentials:

$$\frac{dF}{dx} = \frac{an\ infinitely\ small\ rise\ in\ F}{an\ infinitely\ small\ run\ in\ x}.$$

Example: Velocity as the rate of change in position x with respect to time t can be expressed as

$$v = \frac{infinitely\ small\ change\ in\ position}{infinitely\ small\ time\ interval} = \frac{dx}{dt}.$$

Remark: Many physical laws are correct only when expressed in terms of differentials.

Example: The formula ($distance$) = ($velocity$) × ($time\ interval$) is true either when the velocity is a constant or in terms of differentials, i.e., ($an\ infinitely\ small\ distance$) = ($velocity$) × ($an\ infinitely\ short\ time\ interval$). This is because in an infinitely short time interval, the velocity can be considered a constant. Thus,

$$dx = \frac{dx}{dt}dt = v(t)dt,$$

which is nothing new but $v(t) = dx/dt$.

Example: The formula ($work$) = ($force$) × ($distance$) is true either when the force is a constant or in terms of differentials, i.e., ($an\ infinitely\ small\ work$) = ($force$) × ($an\ infinitely\ small\ distance$). This is because in an infinitely small distance, the force can be considered a constant. Thus,

$$dW = fdx = f\frac{dx}{dt}dt = f(t)v(t)dt,$$

which simply implies: (1) $f = dW/dx$, i.e., force f is nothing but the rate of change in work W with respect to distance x; (2) $dW/dt = f(t)v(t)$, i.e., the rate of change in work W with respect to time t is equal to the product between $f(t)$ and $v(t)$.

Example: The formula ($mass$) = ($density$) × ($volume$) is true either when the density is constant or in terms of differentials, i.e., ($an\ infinitely\ small\ mass$) = ($density$) ×

(*volume of an infinitely small volume*). This is because in an infinitely small piece of volume, the density can be considered a constant. Thus,

$$dm = \rho dV,$$

which implies that $\rho = dm/dV$, i.e., density is nothing but the rate of change in mass with respect to volume.

6 The Chain Rule in Terms of Differentials

When we differentiate a composite function, we need to use the Chain Rule. For example, $f(x) = e^{x^2}$ is a composite function. This is because f is not an exponential function of x but it is an exponential function of $u = x^2$ which is itself a power function of x. Thus,

$$\frac{df}{dx} = \frac{de^{x^2}}{dx} = \frac{de^u}{dx}\frac{du}{du} = (\frac{de^u}{du})(\frac{du}{dx}) = e^u(x^2)' = 2xe^{x^2},$$

where a substitution $u = x^2$ was used to change $f(x)$ into a true exponential function $f(u)$. Therefore, the Chain Rule can be simply interpreted as *the quotient between df and dx is equal to the quotient between df and du multiplied by the quotient between du and dx.* Or simply, divide df/dx by du and then multiply it by du.

However, if we simply regard x^2 as a function different from x, the actual substitution $u = x^2$ becomes unnecessary. Instead, the above derivative can be expressed as

$$\frac{de^{x^2}}{dx} = \frac{de^{x^2}}{dx}\frac{dx^2}{dx^2} = (\frac{de^{x^2}}{dx^2})(\frac{dx^2}{dx}) = e^{x^2}(x^2)' = 2xe^{x^2}.$$

Generally, if $f = f(g(x))$ is a differentiable function of g and g is a differentiable function of x, then

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx},$$

where df, dg, and dx are, respectively, the differentials of the functions f, g, and x.

Example: Calculate df/dx for $f(x) = \sin(\ln(x^2 + e^x))$.

Solution: f is a composite function of another composite function!

$$\frac{df}{dx} = \frac{d\sin(\ln(x^2+e^x))}{d\ln(x^2+e^x)}\frac{d\ln(x^2+e^x)}{d(x^2+e^x)}\frac{d(x^2+e^x)}{dx} = cos(\ln(x^2+e^x))\frac{1}{(x^2+e^x)}(2x+e^x).$$

7 The Product Rule in Terms of Differentials

The Product Rule says if both $u = u(x)$ and $v = v(x)$ are differentiable functions of x, then

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

Multiply both sides by the differential dx, we obtain

$$d(uv) = vdu + udv,$$

which is the Product Rule in terms of differentials.

Example: Let $u = x^2$ and $v = e^{\sin(x^2)}$, then

$$d(uv) = vdu + udv = e^{\sin(x^2)}d(x^2) + x^2 d(e^{\sin(x^2)})$$

$$= 2xe^{\sin(x^2)}dx + x^2e^{\sin(x^2)}\cos(x^2)(2x)dx = 2xe^{\sin(x^2)}[1 + x^2\cos(x^2)]dx.$$

8 Other Properties of Differentials

1. For any differentiable function $F(x)$, $dF = F'(x)dx$ (Recall that $F'(x) = dF/dx$!).
2. For any constant C, $dC = 0$.
3. for any constant C and differentiable function $F(x)$, $d(CF) = CdF = CF'(x)dx$.
4. For any differentiable functions u and v, $d(u \pm v) = du \pm dv$.

9 The Fundamental Theorem in Terms of Differentials

Fundamental Theorem of Calculus: If $F(x)$ is one antiderivative of the function $f(x)$, i.e., $F'(x) = f(x)$, then

$$\int f(x)dx = \int F'(x)dx = \int dF = F(x) + C.$$

Thus, **the integral of the differential of a function F is equal to the function itself plus an arbitrary constant**. This is simply saying that differential and integral are inverse math operations of each other. If we first differentiate a function $F(x)$ and then integrate the derivative $F'(x) = f(x)$, we obtain $F(x)$ itself plus an arbitrary constant. The opposite also is true. If we first integrate a function $f(x)$ and then differentiate the resulting integral $F(x)+C$, we obtain $F'(x) = f(x)$ itself.

Example:

$$\int x^5 dx = \int d(\frac{x^6}{6}) = \frac{x^6}{6} + C.$$

Example:

$$\int e^{-x} dx = \int d(-e^{-x}) = -e^{-x} + C.$$

Example:

$$\int cos(3x)dx = \int d(\frac{sin(3x)}{3}) = \frac{sin(3x)}{3} + C.$$

Example:

$$\int sec^2 x dx = \int dtanx = tanx + C.$$

Example:

$$\int cosh(3x)dx = \int d(\frac{sinh(3x)}{3}) = \frac{sinh(3x)}{3} + C.$$

Example:

$$\int \frac{1}{1+x^2}dx = \int dtan^{-1}x = tan^{-1} + C.$$

Example:

$$\int \frac{1}{1-tanh^2x}dx = \int \frac{1}{sech^2x}dx = \int cosh^2xdx$$

$$= \int \frac{1+cosh(2x)}{2}dx = \frac{1}{2}\int d(x + \frac{sinh(2x)}{2}) = \frac{1}{2}[x + \frac{sinh(2x)}{2}] + C.$$

For more indefinite integrals involving elementary functions, look at the first page of the table of integrals provided at the end of the notes.

10 Integration by Subsitution

Substitution is a necessity when integrating a composite function since we cannot write down the antiderivative of a composite function in a straightforward manner.

Many students find it difficult to figure out the substitution since for different functions the subsitutions are also different. However, there is a general rule in substitution, namely, to change the composite function into a simple, elementary function.

Example: $\int(sin(\sqrt{x})/\sqrt{x})dx$.

Solution: Note that $sin(\sqrt{x})$ is not an elementary sine function but a composite function. The first goal in solving this integral is to change $sin(\sqrt{x})$ into an elementary sine function through substitution. Once you realize this, $u = \sqrt{x}$ is an obvious subsitution. Thus, $du = u'dx = \frac{1}{2\sqrt{x}}dx$, or $dx = 2\sqrt{x}du = 2udu$. Substitute into the integral, we obtain

$$\int \frac{sin(\sqrt{x})}{\sqrt{x}}dx = \int \frac{sin(u)}{u}2udu = 2\int sin(u)du = -2\int dcos(u) = -2cos(u)+C = -2cos(\sqrt{x})+C.$$

Once you become more experienced with subsitutions and differentials, you do not need to do the actual substitution but only symbolically. Note that $x = (\sqrt{x})^2$,

$$\int \frac{sin(\sqrt{x})}{\sqrt{x}}dx = \int \frac{sin(\sqrt{x})}{\sqrt{x}}d(\sqrt{x})^2 = \int \frac{sin(\sqrt{x})}{\sqrt{x}}2\sqrt{x}d\sqrt{x} = 2\int sin(\sqrt{x})d\sqrt{x} = -2cos(\sqrt{x})+C.$$

Thus, as soon as you realize that $\sqrt{x}$ is the substitution, your goal is to change the differential in the integral dx into the differential of $\sqrt{x}$ which is $d\sqrt{x}$.

If you feel that you cannot do it without the actual substitution, that is fine. You can always do the actual substitution. I here simply want to teach you a way that actual subsitution is not a necessity!

Example: $\int x^4 cos(x^5)dx$.

Solution: Note that $cos(x^5)$ is a composite function that becomes a simple cosine function only if the subsitution $u = x^5$ is made. Since $du = u'dx = 5x^4dx$, $x^4dx = \frac{1}{5}du$. Thus,

$$\int x^4 cos(x^5)dx = \frac{1}{5}\int cos(u)du = \frac{sin(u)}{5} + C = \frac{sin(x^5)}{5} + C.$$

Or alternatively,

$$\int x^4 cos(x^5)dx = \frac{1}{5}\int cos(x^5)dx^5 = \frac{1}{5}sin(x^5) + C.$$

Example: $\int x\sqrt{x^2+1}dx$.

Solution: Note that $\sqrt{x^2+1} = (x^2+1)^{1/2}$ is a composite function. We realize that $u = x^2+1$ is a substitution. $du = u'dx = 2xdx$ implies $xdx = \frac{1}{2}du$. Thus,

$$\int x\sqrt{x^2+1}dx = \frac{1}{2}\int \sqrt{u}du = \frac{1}{2}\frac{2}{3}u^{3/2} + C = \frac{(x^2+1)^{3/2}}{3} + C.$$

Or alternatively,

$$\int x\sqrt{x^2+1}dx = \frac{1}{2}\int \sqrt{x^2+1}d(x^2+1) = \frac{1}{2}\frac{2}{3}(x^2+1)^{3/2} + C = \frac{(x^2+1)^{3/2}}{3} + C.$$

In many cases, substitution is required even no obvious composite function is involved.

Example: $\int(\ln x/x)dx$ $(x > 0)$.

Solution: The integrand $\ln x/x$ is not a composite function. Nevertheless, its antiderivative is not obvious to calculate. We need to figure out that $(1/x)dx = d\ln x$, thus by introducing the substitution $u = \ln x$, we obtained a differential of the function $\ln x$ which also appears in the integrand. Therefore,

$$\int(\ln x/x)dx = \int udu = \frac{u^2}{2} + C = \frac{1}{2}(\ln x)^2 + C.$$

It is more natural to consider this substitution is an attempt to change the differential dx into something that is identical to a function that appears in the integrand, namely $d\ln x$. Thus,

$$\int(\ln x/x)dx = \int \ln x d\ln x = \frac{1}{2}(\ln x)^2 + C.$$

Example: $\int \tan x dx$.

Solution: $\tan x$ is not a composite function. Nevertheless, it is not obvious to figure out $\tan x$ is the derivative of what function. However, if we write $\tan x = \sin x/\cos x$, we can regard $1/\cos x$ as a composite function. We see that $u = cosx$ is a candidate for substitution and $du = u'dx = -sin(x)dx$. Thus,

$$\int tanxdx = \int \frac{sinx}{cosx}dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|cosx| + C.$$

Or alternatively,

$$\int tanxdx = \int \frac{sinx}{cosx}dx = -\int \frac{dcosx}{cosx} = -\ln|cosx| + C.$$

Some substitutions are standard in solving specific types of integrals.

Example: Integrands of the type $(a^2 - x^2)^{\pm 1/2}$.

In this case both $x = asinu$ and $x = acosu$ will be good. $x = a\tanh u$ also works $(1-tanh^2u = sech^2u)$. Let's pick $x = asinu$ in this example. If you ask how can we find out that $x = asinu$

is the substitution, the answer is $a^2 - x^2 = a^2(1 - sin^2 u) = a^2 cos^2 u$. This will help us eliminate the half power in the integrand. Note that with this substitution, $u = sin^{-1}(x/a)$, $sinu = x/a$, and $cosu = \sqrt{1 - x^2/a^2}$. Thus,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{d(asinu)}{\sqrt{a^2 - a^2 sin^2 u}} = \int \frac{acosudu}{acosu} = \int du = u + C = sin^{-1}\frac{x}{a} + C.$$

Note that $\int dx/\sqrt{1 - x^2} = sin^{-1}x + C$, we can make a simple substitution $x = au$, thus

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{d(au)}{\sqrt{a^2 - a^2 u^2}} = \int \frac{du}{\sqrt{1 - u^2}} = sin^{-1}u + C = sin^{-1}\frac{x}{a} + C.$$

Similarly,

$$\int \sqrt{a^2 - x^2}dx = \int \sqrt{a^2 - a^2 sin^2 u}d(asinu) = a^2 \int cos^2 udu = a^2 \int \frac{1 + cos(2u)}{2}du$$

$$= \frac{a^2}{2}[u + \frac{sin(2u)}{2}] + C = \frac{a^2}{2}[u + sinucosu] + C = \frac{1}{2}[a^2 sin^{-1}(\frac{x}{a}) + x\sqrt{a^2 - x^2}] + C.$$

Example: Integrands of the type $(a^2 + x^2)^{\pm 1/2}$.

$x = asinh(u)$ is a good substitution since $a^2 + x^2 = a^2 + a^2 sinh^2(u) = a^2(1 + sinh^2(u)) = a^2 cosh^2(u)$, where the hyperbolic identity $1 + sinh^2(u) = cosh^2(u)$ was used. ($x = a\tan u$ is also good since $1 + tan^2 u = sec^2 u$!). Thus,

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \int \frac{dasinh(u)}{\sqrt{a^2 + a^2 sinh^2(u)}} = \int \frac{acosh(u)du}{acosh(u)} = \int du = u + C = sinh^{-1}(\frac{x}{a}) + C$$

$$= \ln|x + \sqrt{a^2 + x^2}| + C,$$

where $sinh^{-1}(x/a) = \ln|x + \sqrt{a^2 + x^2}| - \ln|a|$ was used. And,

$$\int \sqrt{a^2 + x^2}dx = \int \sqrt{a^2 + a^2 sinh^2(u)}dasinh(u) = a^2 \int cosh^2(u)du = \frac{a^2}{2}\int [1 + cosh(2u)]du$$

$$= \frac{a^2}{2}[u + \frac{sinh(2u)}{2}] + C = \frac{1}{2}[a^2 sinh^{-1}(\frac{x}{a}) + a^2 sinh(u)cosh(u)] + C$$

$$= \frac{1}{2}[a^2 \ln|x + \sqrt{a^2 + x^2}| + x\sqrt{a^2 + x^2}] + C,$$

where the following hyperbolic identities were used: $sinh(2u) = 2sinh(u)cosh(u)$, $sinh^{-1}(x/a) = \ln|x + \sqrt{a^2 + x^2}| - \ln|a|$, and $acosh(u) = \sqrt{a^2 + x^2}$.

Example: Integrands of the type $(x^2 - a^2)^{\pm 1/2}$.

$x = acosh(u)$ is prefered since $x^2 - a^2 = a^2 scos^2(u) - a^2 = a^2[cosh^2(u) - 1] = a^2 sinh^2(u)$, where the hyperbolic identity $cosh^2(u) - 1 = sinh^2(u)$ was used. ($x = a \sec u$ is also good since $sec^2 u - 1 = tan^2 u!$). Thus,

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{da\cosh(u)}{\sqrt{a^2\cosh^2(u) - a^2}} = \int \frac{a\sinh(u)du}{a\sinh(u)} = \int du = u + C$$

$$= cosh^{-1}(\frac{x}{a}) + C = \ln|x + \sqrt{x^2 - a^2}| + C.$$

Similarly

$$\int \sqrt{x^2 - a^2}dx = \int \sqrt{a^2\cosh^2(u) - a^2}da\cosh(u) = a^2 \int \sinh^2 udu = \frac{a^2}{2}\int [cosh(2u) - 1]du$$

$$= \frac{1}{2}[x\sqrt{x^2 - a^2} - a^2 \ln|x + \sqrt{x^2 - a^2}|] + C,$$

where $\cosh u = x/a$, $\sinh u = \sqrt{x^2/a^2 - 1}$, and $u = \ln|x + \sqrt{x^2 - a^2}| - \ln a$ were used.

For more details on hyperbolic functions, read the last page of the table of integrals at the end of the notes.

More Exercises on Substitution:

1. Substitution aimed at eliminating a composite function

Example:
(i)

$$\int \frac{4x}{\sqrt[3]{2x^2+3}}dx = \int \frac{d(2x^2+3)}{\sqrt[3]{2x^2+3}} = \int u^{-1/3}du = \frac{3}{2}u^{2/3} + C = \frac{3}{2}(2x^2+3)^{2/3} + C.$$

(ii)

$$\int \frac{x}{\sqrt{2x+3}}dx = \frac{1}{4}\int \frac{2x+3-3}{\sqrt{2x+3}}d(2x+3) = \frac{1}{4}\int \frac{u-3}{u^{1/2}}du = \frac{1}{4}\int [u^{1/2} - 3u^{-1/2}]du$$

$$= \frac{1}{4}[\frac{2}{3}u^{3/2} - 6u^{1/2}] + C = \frac{1}{4}[\frac{2}{3}(2x+3)^{3/2} - 6(2x+3)^{1/2}] + C.$$

Edwards/Penney $5^{th}-ed$ 5.7 Problems.

(1) $\int (3x-5)^{17}dx$ (3) $\int x\sqrt{x^2+9}dx$ (4) $\int \frac{x^2}{\sqrt[3]{2x^3-1}}dx$

(7) $\int x\sin(2x^2)dx$ (9) $\int (1-cosx)^5 sinxdx$ (15) $\int \frac{dx}{\sqrt{7x+5}}dx$

(23) $\int x\sqrt{2-3x^2}dx$ (31) $\int cos^3xsinxdx$ (33) $\int tan^3xsec^2xdx$

(45) $\int \frac{cos(\sqrt{x})}{\sqrt{x}}dx$ (49) $\int_0^{\pi/2} (1+3sin\theta)^{3/2}cos\theta d\theta$ (51) $\int_0^{\pi/2} e^{sinx}cosxdx$

Edwards/Penney $5^{th}-ed$ 9.2 Problems.

(7) $\int \frac{cot(\sqrt{y})csc(\sqrt{y})}{\sqrt{y}}dy$ (11) $\int e^{-cotx}csc^2xdx$ (13) $\int \frac{(\ln t)^10}{t}dt,\ (t>0)$

(21) $\int tan^4(3x)sec^2(3x)dx$ (25) $\int \frac{(1+\sqrt{x})^4}{\sqrt{x}}dx$ (31) $\int x^2\sqrt{x+2}dx.$

2. Substitution to achieve a function in differential that appears in the integrand

Example:
(i)
$$\int \frac{e^x}{1+e^{2x}}dx = \int \frac{de^x}{1+(e^x)^2} = \int \frac{du}{1+u^2} = tan^{-1}(u) + C = tan^{-1}(e^x) + C.$$

(ii)
$$\int \frac{x}{\sqrt{e^{2x^2}-1}}dx = \frac{1}{2}\int \frac{dx^2}{\sqrt{e^{2x^2}(1-e^{-2x^2})}} = \frac{1}{2}\int \frac{du}{\sqrt{e^{2u}(1-e^{-2u})}} = \frac{1}{2}\int \frac{du}{e^u\sqrt{1-(e^{-u})^2}}$$

$$= -\frac{1}{2}\int \frac{de^{-u}}{\sqrt{1-(e^{-u})^2}} = -\frac{1}{2}\int \frac{dy}{\sqrt{1-y^2}} = \frac{1}{2}cos^{-1}(y)+C = \frac{1}{2}cos^{-1}(e^{-u})+C = \frac{1}{2}cos^{-1}(e^{-x^2})+C.$$

Edwards/Penney $5^{th} - ed$ 9.2 Problems (difficult ones!).

(17) $\int \frac{e^{2x}}{1+e^{4x}}dy,\ (u = e^{2x})$ (19) $\int \frac{3x}{\sqrt{1-x^4}}dx,\ (u = x^2)$ (23) $\int \frac{cos\theta}{1+sin^2\theta}d\theta,\ (u = sin\theta)$

(27) $\int \frac{1}{(1+t^2)tan^{-1}t}dt,\ (u = tan^{-1}t)$ (29) $\int \frac{1}{\sqrt{e^{2x}-1}}dx,\ (u = e^{-x})$

3. Special Trigonometric Substitutions

Example:
(i)
$$\int \frac{x^3}{\sqrt{1-x^2}}dx = \int \frac{\sin^3 u d\sin u}{\sqrt{1-\sin^2 u}} = \int \sin^3 u du = -\int (1-\cos^2 u) d\cos u$$

$$= \frac{cos^3 u}{3} - \cos u + C = \frac{(1-x^2)^{3/2}}{3} - \sqrt{1-x^2} + C.$$

Edwards/Penney $5^{th} - ed$ 9.6 Problems (difficult ones!).

(1) $\int \frac{1}{\sqrt{16-x^2}}dx,\ (x = 4\sin u)$ (9) $\int \frac{\sqrt{x^2-1}}{x}dx,\ (x = \cosh u)$ (11) $\int x^3\sqrt{9+4x^2}dx,\ (2x = 3\sinh u)$

(13) $\int \frac{\sqrt{1-4x^2}}{x}dx$, $(2x = \sin u)$ (19) $\int \frac{x^2}{\sqrt{1+x^2}}dx$, $(x = \sinh u)$ (27) $\int \sqrt{9+16x^2}dx$, $(4x = 3\sinh u)$

11 Integration by Parts

Integration by Parts is the integral version of the Product Rule in differentiation. The Product Rule in terms of differentials reads,

$$d(uv) = vdu + udv.$$

Integrating both sides, we obtain

$$\int d(uv) = \int vdu + \int udv.$$

Note that $\int d(uv) = uv + C$, the above equation can be expressed in the following form,

$$\int udv = uv - \int vdu.$$

Generally speaking, we need to use Integration by Parts to solve many integrals that involve the product between two functions. In many cases, Integration by Parts is most efficient in solving integrals of the product between a polynomial and an exponential, a logarithmic, or a trigonometric function. It also applies to the product between exponential and trigonometric functions.

Example: $\int xe^x dx$.

Solution: In order to eliminate the power function x, we note that $(x)' = 1$. Thus,

$$\int xe^x dx = \int xde^x = xe^x - \int e^x dx = xe^x - e^x + C.$$

Example: $\int x^2 cosx dx$.

Solution: In order to eliminate the power function x^2, we note that $(x^2)'' = 2$. Thus, we need to use Integration by Patrs twice.

$$\int x^2 cosx dx = \int x^2 dsinx = x^2 sinx - \int sinx dx^2 = x^2 sinx - \int 2x d(-cosx)$$

$$= x^2 sinx + 2\int x dcosx = x^2 sinx + 2x cosx - 2\int cosx dx = x^2 sinx + 2x cosx - 2sinx + C.$$

Example: $\int x^2 e^{-2x} dx$.

Solution: In order to eliminate the power function x^2, we note that $(x^2)'' = 2$. Thus, we need to use Integration by Patrs twice. However, the number (-2) can prove extremely annoying and easily cause errors. Here is how we use substitution to avoid this problem.

$$\int x^2 e^{-2x} dx = \frac{-1}{2^3}\int (-2x)^2 e^{-2x} d(-2x) = \frac{-1}{8}\int u^2 e^u du = \frac{-1}{8}[u^2 e^u - 2\int u e^u du]$$

$$= \frac{-1}{8}[u^2 e^u - 2(ue^u - e^u)] + C = \frac{-e^u}{8}[u^2 - 2u + 2] + C = \frac{-e^{-2x}}{8}[4x^2 + 4x + 2] + C.$$

When integrating the product between a polynomial and a logarithmic function, the main goal is to eliminate the logarithmic function by differentiating it. This is because $(\ln x)' = 1/x$.

Example: $\int \ln x dx$, $(x > 0)$.

Solution:

$$\int \ln x dx = x\ln x - \int x d\ln x = x\ln x - \int x\frac{dx}{x} = x\ln x - x + C.$$

Example: $\int x(\ln x)^2 dx$, $(x > 0)$.

Solution:

$$\int x(\ln x)^2 dx = \frac{1}{2}\int (\ln x)^2 dx^2 = \frac{1}{2}[x^2(\ln x)^2 - \int x^2(2\ln x)\frac{dx}{x}] = \frac{1}{2}[x^2(\ln x)^2 - \int \ln x dx^2]$$

$$= \frac{1}{2}[x^2(\ln x)^2 - x^2\ln x + \frac{x^2}{2}] + C = \frac{x^2}{2}[(\ln x)^2 - \ln x + \frac{1}{2}] + C.$$

More Exercises on Integration by Parts:

Example:
(i)

$$\int t\sin t dt = -\int t d\cos t = -[t\cos t - \int \cos t dt] = [\sin t - t\cos t] + C.$$

(ii)

$$\int tan^{-1}x dx = xtan^{-1}x - \int x dtan^{-1}x = xtan^{-1}x - \int \frac{x}{1+x^2}dx$$

$$= xtan^{-1}x - \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2} = xtan^{-1}x - \ln\sqrt{1+x^2} + C.$$

Edwards/Penney $5^{th} - ed$ 9.3 Problems.

(1) $\int xe^{2x}dx$ (5) $\int x\cos(3x)dx$ (7) $\int x^3\ln x dx,\ (x > 0)$

(11) $\int \sqrt{y}\ln y dy$ (13) $\int (\ln t)^2 dt$ (19) $\int csc^3\theta d\theta$

(21) $\int x^2 tan^{-1}x dx$ (27) $\int x csc^2 x dx$ (19) $\int csc^3\theta d\theta$

12 Integration by Partial Fractions

Rational functions are defined as the quotient between two polynomials:

$$R(x) = \frac{P_n(x)}{Q_m(x)}$$

where $P_n(x)$ and $Q_m(x)$ are polynomials of degree n and m respectively. The method of partial fractions is an **algebraic** technique that decomposes $R(x)$ into a sum of terms:

$$R(x) = \frac{P_n(x)}{Q_m(x)} = p(x) + F_1(x) + F_2(x) + \cdots F_k(x),$$

where $p(x)$ is a polynomial and $F_i(x)$, $(i = 1, 2, \cdots, k)$ are fractions that can be integrated easily.

The method of partial fractions is an area many students find very difficult to learn. It is related to **algebraic** techniques that many students have not been trained to use. The most typical claim is that there is no fixed formula to use. It is not our goal in this course to cover this topics in great details (Read Edwards/Penney for more details). Here, we only study two simple cases.

Case I: $Q_m(x)$ is a power function, i.e., $Q_m(x) = (x-a)^m$ ($Q_m(x) = x^m$ if $a = 0$!).

Example:

$$\int \frac{2x^2 - x + 3}{x^3} dx.$$

This integral involves the simplest partial fractions:

$$\frac{A+B+C}{D} = \frac{A}{D} + \frac{B}{D} + \frac{C}{D}.$$

Some may feel that it is easier to write the fractions in the following form: $D^{-1}(A+B+C) = D^{-1}A + D^{-1}B + D^{-1}C$. Thus,

$$\int \frac{2x^2 - x + 3}{x^3} dx = \int [\frac{2x^2}{x^3} - \frac{x}{x^3} + \frac{3}{x^3}] dx = \int [\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x^3}] dx = \ln x^2 + \frac{1}{x} - \frac{3}{2x^2} + C.$$

Example:

$$\int \frac{2x^2 - x + 3}{x^2 - 2x + 1} dx = \int \frac{2x^2 - x + 3}{(x-1)^2} dx = \int \frac{2(x-1+1)^2 - (x-1+1) + 3}{(x-1)^2} dx$$

$$= \int \frac{2(x-1)^2 + 4(x-1) + 2 - (x-1) - 1 + 3}{(x-1)^2} dx = \int \frac{2(x-1)^2 + 3(x-1) + 4}{(x-1)^2} dx$$

$$= \int [2 + \frac{3}{x-1} + \frac{4}{(x-1)^2}] d(x-1) = 2(x-1) + 3\ln|x-1| - \frac{4}{x-1} + C.$$

Case II: $P_n(x) = A$ is a constant and $Q_m(x)$ can be factorized into the form $Q_2(x) = (x-a)(x-b)$, $Q_3(x) = (x-a)(x-b)(x-c)$, or $Q_m(x) = (x-a_1)(x-a_2)\cdots(x-a_m)$.

Example:

$$I = \int \frac{2}{x^2-2x-3}dx = \int \frac{2dx}{(x-3)(x+1)} = 2\int [\frac{A}{x-3} + \frac{B}{x+1}]dx.$$

Since $\frac{A}{x-3} + \frac{B}{x+1} = \frac{1}{(x-3)(x+1)}$ implies that $A(x+1) + B(x-3) = 1$. Setting $x = 3$ in this equation, we obtain $A = 1/4$. Setting $x = -1$, we obtain $B = -1/4$. Thus,

$$I = 2\int [\frac{A}{x-3} + \frac{B}{x+1}]dx = 2\int [\frac{1/4}{x-3} + \frac{-1/4}{x+1}]dx = \frac{1}{2}\int [\frac{dx}{x-3} - \frac{dx}{x+1}] = \frac{1}{2}\ln|\frac{x-3}{x+1}| + C.$$

Example:

$$I = \int \frac{dx}{(x-1)(x-2)(x-3)} = \int [\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}]dx.$$

Since $\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} = 1$ implies that $A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) = 1$. Setting $x = 1$ in this equation, we obtain $A = 1/2$. Setting $x = 2$, we obtain $B = -1$. Setting $x = 3$, we get $C = 1/2$. Thus,

$$I = \int [\frac{1/2}{x-1} + \frac{-1}{x-2} + \frac{1/2}{x-3}]dx = \frac{1}{2}\ln\frac{|(x-1)(x-3)|}{(x-2)^2} + C.$$

More Exercises on Integration by Partial Fractions:

Examples:
(i)

$$\int \frac{dx}{x^2-3x} = \int \frac{dx}{x(x-3)} = \frac{1}{3}\int [\frac{1}{x-3} - \frac{1}{x}]dx = \frac{1}{3}\ln|\frac{x-3}{x}| + C.$$

(ii)

$$\int \frac{dx}{x^2+x-6} = \int \frac{dx}{(x+3)(x-2)} = \frac{1}{5}\int [\frac{1}{x-2} - \frac{1}{x+3}]dx = \frac{1}{5}\ln|\frac{x-2}{x+3}| + C.$$

(iii)

$$\int \frac{x-1}{x+1}dx = \int \frac{x+1-2}{x+1}dx = \int [1 - \frac{2}{x+1}]dx = x - \ln(x+1)^2 + C.$$

Edwards/Penney $5^{th} - ed$ 9.5 Problems.

(Li) $\int \frac{2x^3+3x^2-5}{x^3}dx$ (15) $\int \frac{dx}{x^2-4}$ (Li) $\int \frac{dx}{x^2-2x-8}$

(Li) $\int \frac{x}{x^2+5x+6}dx$, $(Hint: x = x+2-2)$ (Li) $\int \frac{x^2}{1+x^2}dx$, $(Hint: x^2 = 1+x^2-1)$

(Li) $\int \frac{x^2}{x^2+x}dx$, $(Hint:\ x^2 = [(x+1)-1]^2)$ (23) $\int \frac{x^2}{(x+1)^3}dx$

13 Integration by Multiple Techniques

Whenever we encounter a complex integral, often the first thing to do is to use substitution to eliminate the composite function(s).

Example: $I = \int x^{-3}e^{1/x}dx.$

Solution: Note that $e^{1/x}$ is a composite fucntion. The substitution to change it into a simple, elementary function is $u = 1/x$, $du = u'dx = -x^{-2}dx$ or $x^{-2}dx = -du = d(-1/x)$. Thus,

$$I = \int x^{-3}e^{1/x}dx = \int (1/x)e^{1/x}d(-1/x) = -\int ue^u du = -[ue^u - e^u] + C = e^{1/x}[1 - 1/x] + C.$$

Example: $I = \int \cos(\sqrt{x})dx$.

Solution: Note that $cos(\sqrt{x})$ is a composite fucntion. The substitution to change it into a simple, elementary function is $u = \sqrt{x}$, $du = u'dx = \frac{1}{2\sqrt{x}}dx$ or $dx = 2\sqrt{x}du = 2udu$. However, note that $x = (\sqrt{x})^2$,

$$I = \int \cos(\sqrt{x})dx = \int \cos(\sqrt{x})d(\sqrt{x})^2 = \int cos(u)du^2 = 2\int ucos(u)du$$

$$= 2\int udsin(u) = 2[usin(u) + cos(u)] + C = 2[\sqrt{x}sin(\sqrt{x}) + cos(\sqrt{x})] + C.$$

Example: $I = \int sec(x)dx = \int (1/cos(x))dx$.

Solution: This is a difficult integral. Note that $1/cos(x)$ is a composite fucntion. One substitution to change it into a simple, elementary function is $u = cos(x)$. However, $x = cos^{-1}(u)$ and $dx = x'du = \frac{-du}{\sqrt{1-u^2}}$. $1/\sqrt{1-u^2}$ is still a composite function that is difficult to deal with. Here is how we can deal with it. It involves trigonometric substitution followed by partial fractions.

$$I = \int \frac{dx}{cos(x)} = \int \frac{cos(x)dx}{cos^2(x)} = \int \frac{dsin(x)}{1 - sin^2(x)} = \int \frac{du}{1-u^2} = -\int [\frac{1}{u-1} - \frac{1}{u+1}]\frac{du}{2}$$

$$= \ln\sqrt{|\frac{u+1}{u-1}|} + C = \ln|\frac{u+1}{\sqrt{1-u^2}}| + C = \ln|\frac{sin(x)+1}{cos(x)}| + C = \ln|tan(x) + sec(x)| + C,$$

where $u = sin(x)$ and $\sqrt{1-u^2} = cos(x)$ were used!

More Exercises on Integration by Multiple techniques:

1. Substitution followed by Integration by Parts

Example:
(i)

$$\int x^3\sin(x^2)dx = \frac{1}{2}\int x^2\sin(x^2)dx^2 = \frac{1}{2}\int u\sin u du = \frac{-1}{2}\int ud\cos u$$

$$= \frac{1}{2}[\sin u - u\cos u] + C = \frac{1}{2}[\sin(x^2) - x^2\cos(x^2)] + C.$$

(ii)

$$\int xtan^{-1}\sqrt{x}dx = \int (\sqrt{x})^2 tan^{-1}\sqrt{x}d(\sqrt{x})^2 = \int u^2 tan^{-1}udu^2 = \frac{1}{2}\int tan^{-1}udu^4$$

$$= \frac{1}{2}[u^4 tan^{-1}u - \int u^4 dtan^{-1}u] = \frac{1}{2}[u^4 tan^{-1}u - \int \frac{u^4-1+1}{1+u^2}du] = \frac{1}{2}[u^4 tan^{-1}u - \int (u^2-1+\frac{1}{1+u^2})du]$$

$$= \frac{1}{2}[u^4 tan^{-1}u - \frac{u^3}{3} + u - tan^{-1}u] + C = \frac{1}{2}[(x^2-1)tan^{-1}\sqrt{x} - \frac{x^{3/2}}{3} + \sqrt{x}] + C.$$

Edwards/Penney $5^{th} - ed$ 9.3 Problems.

(15) $\int x\sqrt{x+3}dx,\ u = x+3$ (17) $\int x^5\sqrt{x^3+1}dx,\ u = x^3+1$

(23) $\int sec^{-1}\sqrt{x}dx,\ u = \sqrt{x}$ (29) $\int x^3\cos(x^2)dx,\ u = x^2$ (25) $\int tan^{-1}\sqrt{x}dx,\ u = \sqrt{x}$

(31) $\int \frac{\ln x}{x\sqrt{x}}dx,\ u = \sqrt{x}$ (37) $\int e^{-\sqrt{x}}dx,\ u = \sqrt{x}$

2. Substitution followed by Integration by Partial Fractions

Example:
(i)

$$\int \frac{\cos x dx}{\sin^2 x - \sin x - 6} = \int \frac{d\sin x}{\sin^2 x - \sin x - 6} = \int \frac{du}{(u-3)(u+2)} = \frac{1}{5}\ln|\frac{u-3}{u+2}| + C = \frac{1}{5}\ln|\frac{\sin x - 3}{\sin x + 2}| + C$$

Edwards/Penney $5^{th} - ed$ 9.5 Problems (diffifult ones!)

(40) $\int \frac{\sec^2 t}{\tan^3 t + tan^2 t}dt$ (37) $\int \frac{e^{4t}}{(e^{2t}-1)^3}dt$ (39) $\int \frac{1+\ln t}{t(3+2\ln t)^2}dt$

2. Integration by Parts followed by Partial Fractions

Example:
(i)

$$\int (2x+2)\tan^{-1} x dx = \int \tan^{-1} x d(x^2+2x) = (x^2+2x)\tan^{-1} x - \int (x^2+2x) d\tan^{-1} x$$

$$= (x^2+2x)\tan^{-1} x - \int \frac{1+x^2+2x-1}{1+x^2} dx = (x^2+2x)\tan^{-1} x - \int [dx + \frac{d(1+x^2)}{1+x^2} - \frac{dx}{1+x^2}]$$

$$= (x^2+2x)\tan^{-1} x - x - \ln(1+x^2) + \tan^{-1} x + C = (x+1)^2 \tan^{-1} x - x - \ln(1+x^2) + C.$$

Li's Problems (diffifult ones!)

(Li) $\int (2t+1)\ln t dt$

14 Applications of Integrals

The central idea in all sorts of application problems involving integrals is *breaking the whole into pieces and then adding the pieces up to obtain the whole.*

14.1 Area under a curve.

We know the area of a rectangle is:

$$Area = height \times width,$$

(see Fig. 1(a)). This formula can not be applied to the area under the curve in Fig. 1(b) because the top side of the area is NOT a straight, horizontal line. However, if we divide the area into infinitely many rectangles with infinitely thin width dx, the infinitely small area of the rectangle located at x is

$$dA = height \times infintely\ small\ width = f(x)dx.$$

Adding up the area of all the thin rectangles, we obtain the area under the curve

$$Area = A(b) - A(a) = \int_0^{A(b)} dA = \int_a^b f(x)dx,$$

where $A(x) = \int_a^x f(s)ds$ is the area under the curve between a and x.

14.2 Volume of a solid.

We know the volume of a cylinder is

$$Volume = (cross - section\ area) \times length,$$

(see Fig. 2(a)). This formula does not apply to the volume of the solid obtained by rotating a curve around the x-axis (see Fig. 2(b)). This is because the cross-section area is NOT a constant but changes along the length of this “cylinder”. However, if we cut this “cylinder” into infinitely many disks of infinitely small thickness dx, the volume of such a disk located at x is

$$dV = cross - section\ area \times infintely\ small\ thickness = \pi r^2 dx = \pi (f(x))^2 dx.$$

Adding up the volume of all such thin disks, we obtain the volume of the solid

$$Volume = V(b) - V(a) = \int_0^{V(b)} dV = \pi \int_a^b (f(x))^2 dx,$$

where $V(x) = \pi \int_a^x (f(s))^2 ds$ is the volume of the “cylinder” between a and x.

14.3 Arc length.

We know the length of the hypotenuse of a right triangle is

$$hypotenuse = \sqrt{base^2 + height^2},$$

(see Fig. 3(a)). This formula does not apply to the length of the curve in Fig. 3(b) because the curve is NOT a straight line and the geometric shape of the area enclosed in solid lines is NOT

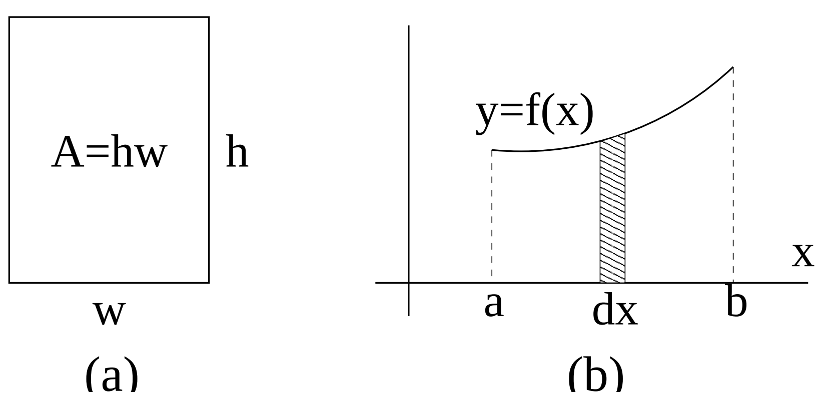

Figure 1: Area under a curve.

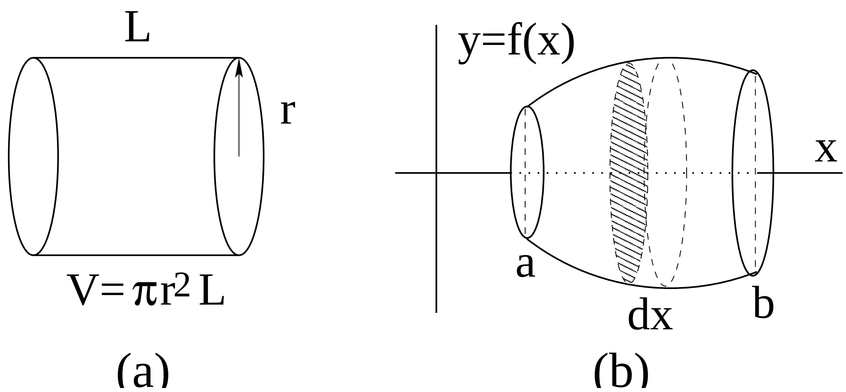

Figure 2: Volume of a solid obtained by rotating a curve.

a right triangle. However, if we "cut" the curve on interval $[a,\ b]$ into infinitely many segments each with a horizontal width dx, each curve segment can be regarded as the hypotenuse of an inifinitely small right triangle with base dx and height dy. Thus, the length of one such curve segments located at x is

$$dl = \sqrt{(infinitely\ small\ base)^2 + (infinitely\ small\ height)^2}$$

$$= \sqrt{dx^2 + dy^2} = \sqrt{[1 + (\frac{dy}{dx})^2]dx^2} = [\sqrt{1 + (y')^2}]dx.$$

The total length of this curve is obtained by adding the lengths of all the segments together.

$$L = \int_0^{l(b)} dl = \int_a^b [\sqrt{1 + (y')^2}]dx,$$

where $l(x) = \int_a^x [\sqrt{1 + [y'(s)]^2}]ds$ is the length of the curve between a and x.

Example: Calculate the length of the curve $y = x^2/2$ on the interval $[-2,\ 2]$.

Solution: Because of the symmetry, we only need to calculate the length on the interval $[0,\ 2]$ and multiply it by two. Thus, noticing that $y' = x$,

$$L = 2\int_{l(0)}^{l(2)} dl = 2\int_0^2 [\sqrt{1 + (y')^2}]dx = 2\int_0^2 [\sqrt{1 + x^2}]dx = 2F(x)|_0^2,$$

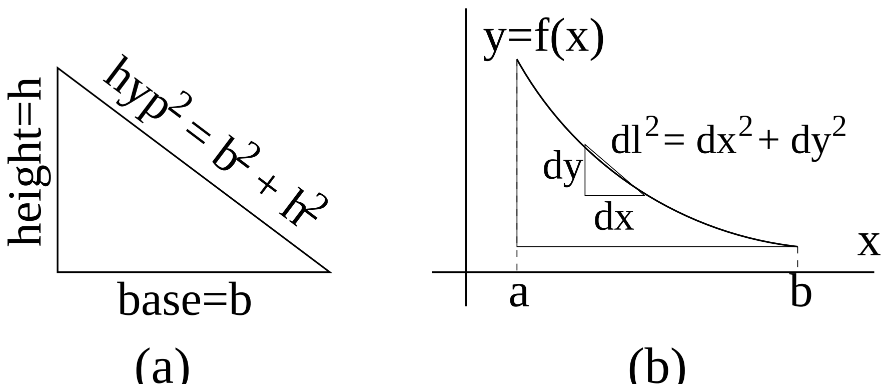

Figure 3: Length of a curve.

where $F(x) = \int [\sqrt{1+x^2}]dx$. Now, this integral can be solved by the standard substitution $x = sinh(u)$. Note that $1 + x^2 = 1 + sinh^2(u) = cosh^2(u)$ and $dx = dsinh(u) = cosh(u)du$, we obtain by subsitution

$$F(x) = \int [\sqrt{1+x^2}]dx = \int cosh(u)cosh(u)du = \int cosh^2(u)du = \frac{1}{2}\int [1 + cosh(2u)]du$$

$$= \frac{1}{2}[u + \frac{1}{2}sinh(2u)] + C = \frac{1}{2}[u + sinh(u)cosh(u)] + C = \frac{1}{2}[\ln|x + \sqrt{1+x^2}| + x\sqrt{1+x^2}] + C,$$

where the inverse substitution $sinh(u) = x$, $cosh(u) = \sqrt{1+x^2}$, and $u = sinh^{-1}(x) = \ln|x + \sqrt{1+x^2}|$ were used in the last step.

Thus,

$$L = 2F(x)|_0^2 = \ln|2 + \sqrt{1+2^2}| + 2\sqrt{1+2^2} = \ln(2+\sqrt{5}) + 2\sqrt{5} \approx 5.916.$$

14.4 From density to mass.

We have learned that the mass contained in a solid is

$$Mass = density \times volume.$$

The formula is no longer valid if the density is NOT a constant but is nonuniformly distributed in the solid. However, if you cut the mass up into infinitely many pieces of inifinitely small volume dV, then the density in this little volume can be considered a constant so that we can apply the above formula. Thus, the infinitely small mass contained in the little volume is

$$dm = density \times (infinitely\ small\ volume) = \rho dV.$$

The total mass is obtained by adding up the masses of all such small pieces.

$$m = \int_0^{m(V(b))} dm = \int_{V(a)}^{V(b)} \rho dV,$$

where $V(x) = \int_0^{V(x)} dV$ is the volume of the portion of the solid between a and x; while $m(V(x)) = \int_0^{m(V(x))} dm$ is the mass contained in the volume $V(x)$.

Example: Calculate the total amount of pollutant in an exhaust pipe filled with polluted liquid which connects a factory to a river. The pipe is 100 m long with a diameter of 1 m. The density of the pollutant in the pipe is $\rho(x) = e^{-x/10}$ (kg/m^3), x is the distance in meters from the factory.

Solution: Since the density only varies as a function of the distance x, we can "cut" the pipe into thin cylindrical disks along the axis of the pipe. Thus, $dV = d(\pi r^2 x) = \pi r^2 dx$. Using the integral form of the formula, we obtain

$$m = \int_{m(0)}^{m(100)} dm = \int_{V(0)}^{V(100)} \rho(x) dV$$

$$= \int_0^{100} \rho(x)\pi r^2 dx = \frac{\pi}{4}\int_0^{100} e^{-x/10} dx = \frac{10\pi}{4}[1 - e^{-10}] \approx 7.85\ (kg).$$

Example: The air density h meters above the earth's surface is $\rho(h) = 1.28e^{-0.000124h}$ (kg/m^3). Find the mass of a cylindrical column of air 4 meters in diameter and 25 kilometers high. (3 points)

Solution: Similar to the previous problem, the density only varies as a function of the altitutde h. Thus, we "cut" this air column into horizontal slices of thin disks with volume $dV = \pi r^2 dh$.

$$m = \int_{m(0)}^{m(25{,}000)} dm = \int_{V(0)}^{V(25{,}000)} \rho dV = \int_0^{25{,}000} \rho(h)\pi r^2 dh = \int_0^{25{,}000} \rho(h)\pi 2^2 dh$$

$$= \frac{4 \times 1.28\pi}{-0.000124} e^{-0.000124h}|_0^{25{,}000} = \frac{4 \times 1.28\pi}{0.000124}[1 - e^{-3.1}] = 123{,}873.71\ (kg).$$

Very often the density is given as mass per unit area (two dimensional density). In this case, $dm = \rho dA$, where dA is an infinitely small area. Similarly, if the density is given as mass per unit length (one dimensional density), then $dm = \rho dx$, where dx is an infinitely small length.

Example: The density of bacteria growing in a circular colony of radius 1 cm is observed to be $\rho(r) = 1 - r^2,\ 0 \leq r \leq 1$ (in units of one million cells per square centimeter), where r is the distance (in cm) from the center of the colony. What is the total number of bacteria in the colony?

Solution: This is a problem involving a two dimensional density. However, the density varies only in the radial direction. Thus, we can divide the circular colony into infinitely many concentric ring areas. We can use the formula $(\#\ of\ cells\ in\ a\ ring) = density \times (area\ of\ the\ ring)$, i.e., $dm = \rho(r)dA$. The area, dA, of a ring of bacteria colony with radius r and width dr is $dA = \pi(r + dr)^2 - \pi r^2 = 2\pi r dr + \pi dr^2 = 2\pi r dr$, where dr^2 is infinitely smaller than dr, thus is ignored as $dr \to 0$. Another way to calculate the area of the ring is to consider the ring as a rectangular area of length $2\pi r$ (i.e., the circumference) and width dr, $dA = length \times width = 2\pi r dr$. Or simply, $A = \pi r^2$, thus $dA = d(\pi r^2) = \pi dr^2 = 2\pi dr$.

$$m = \int_{m(0)}^{m(1)} dm = \int_{A(0)}^{A(1)} \rho dA = \int_0^1 \rho(r) d(\pi r^2) = \int_0^1 \rho(r) 2\pi r dr$$

$$= \pi \int_0^1 (1 - r^2) dr^2 = -\pi \frac{(1 - r^2)^2}{2}|_0^1 = \frac{\pi}{2}\ (millions).$$

Example: The density of a band of protein along a one-dimensional strip of gel in an electrophoresis experiment is given by $\rho(x) = 6(x-1)(2-x)$ for $1 \leq x \leq 2$, where x is the distance along the strip in cm and $\rho(x)$ is the protein density (i.e. protein mass per cm) at distance x. Find the total mass of the protein in the band for $1 \leq x \leq 2$.

Solution: This is a problem involving a one dimensional density. Thus, $(infinitely\ small\ mass) = density \times (infinitely\ small\ length)$, i.e., $dm = \rho(x)dx$.

$$m = \int_{m(1)}^{m(2)} dm = \int_1^2 \rho(x)dx = -6\int_1^2 (x-1)(x-2)dx = -[2x^3-9x^2+12x]\Big|_1^2 = -4-(-5) = 1.$$

14.5 Work done by a varying force.

Elementary physics tells us that work done by a force is

$$Work = force \times distance.$$

The formula is no longer valid if the force is NOT a constant over the distance covered but changes. However, if we "cut" the distance into infinitely many segments of inifinitely short distances dx, then the force on that little distance can be considered a constant. Thus, the infinitely small work done on a distance dx is

$$dW = force \times (infintely\ small\ distance) = fdx.$$

The total work is obtained by adding up the work done on all such small distances.

$$W = \int_{W(a)}^{W(b)} dW = \int_a^b fdx.$$

Example: Calculate total work done by the earth's gravitational force on a satellite of mass m when it is being launched into the space. This is the lower limit of energy required to launch it into space. Note that the gravitational force experienced by a mass m located at a distance r from the center of the earth is: $f(r) = GM_e m/r^2$, where G is the universal gravitational constant and M_e the mass of the earth. Assume that the earth is a sphere with radius R_e and that $r = \infty$ when the satellite is "outside" the earth's gravitational field. (Consider G, M_e, R_e, and m are known constants.)

Solution: We know that work done on an infinitely small distance dr when the satellite is located at a distance r from the center of the earth is: $dW = f(r)dr = (GM_e m/r^2)dr$. Thus,

$$W = \int_{W(R_e)}^{W(\infty)} dW = \int_{R_e}^{\infty} f(r)dr = GM_e m\int_{R_e}^{\infty} \frac{1}{r^2}dr.$$

Integrals that involve a limit which is ∞ are called *improper integrals*. We shall study improper integrals in more detail later. In many cases, we can just treat ∞ as a normal upper limit of this definite integral.

$$W = GM_e m \int_{R_e}^{\infty} \frac{1}{r^2} dr = GM_e m(-\frac{1}{r})|_{R_e}^{\infty} = \frac{GM_e m}{R_e} = mgR_e,$$

where the gravitaional constant on the surface of the earth $g = GM_e/R_e^2$ was used.

Example: For the time interval $0 \leq t \leq 10$, write down the definite integral determining the total work done by a force, $f(t)$ ($Newton$), experienced by a particle moving on a straight line with a speed, $v(t) > 0$ (m/s). $f(t)$ and $v(t)$ always point to the same direction and both change with time t (s).

Solution:

$$W = \int_{W(0)}^{W(10)} dW = \int_{x(0)}^{x(10)} f(t)dx = \int_0^{10} f(t)(\frac{dx}{dt})dt = \int_0^{10} f(t)v(t)dt.$$

15 Elementary integrals

All of these follow immediately from the table of derivatives. They should be memorized.

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [\alpha f + \beta g]\,dx = \alpha\int f\,dx + \beta\int g\,dx$$
$$\int c\,dx = cx + C \qquad \int dx = x + C$$
$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$
$$\int e^x\,dx = e^x + C \qquad \int \ln x\,dx = x\ln x - x + C$$

The following are elementary integrals involving trigonometric functions.

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$
$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}tan^{-1}\frac{x}{a} + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$
$$\int \tan x\;dx = -\ln|\cos x| + C \qquad \int \cot x\;dx = \ln|\sin x| + C$$
$$\int \sec x\;dx = \ln|\tan x + \sec x| + C \qquad \int \csc x\;dx = \ln|\cot x - \csc x| + C$$

The following are elementary integrals involving hyperbolic functions (for details read the Section on Hyperbolic Functions).

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$
$$\int \mathrm{sech}^2 x\,dx = \tanh x + C \qquad \int \mathrm{csch}^2 x\,dx = -\coth x + C$$
$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C \qquad \int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}\frac{x}{a} + C$$
$$\int \tanh x\;dx = \ln(\cosh x) + C \qquad \int \coth x\;dx = \ln|\sinh x| + C$$
$$\int \mathrm{sech}\,x\;dx = tan^{-1}(\sinh x) + C \qquad \int \mathrm{csch}\,x\;dx = \ln|\coth x - \mathrm{csch}\,x| + C$$

16 A selection of more complicated integrals

These begin with the two basic formulas, change of variables and integration by parts.

$\int f(g(x))g'(x)\,dx = \int f(u)\,du$ where $u = g(x)$ (change of variables)

$\int f(g(x))\,dx = \int f(u)\frac{dx}{du}\,du$ where $u = g(x)$ (different form of the same change of variables)

$\int e^{cx}\,dx = \frac{1}{c}e^{cx} + C \quad (c \neq 0)$

$\int a^x\,dx = \frac{1}{\ln a}a^x + C$ (for $a > 0,\ a \neq 1$)

$\int \ln x\,dx = x\ln x - x + C$

$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C,\ a \neq 0$

$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C,\ a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a} + C,\ a > 0$

$\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \ln|x + \sqrt{x^2\pm a^2}| + C$

To compute $\int \frac{1}{x^2+bx+c}\,dx$ we complete the square

$$x^2+bx+c = x^2+bx+\frac{b^2}{4}+c-\frac{b^2}{4} = \left(x+\frac{b}{2}\right)^2 + c - \frac{b^2}{4}$$

If $c - b^2/4 > 0$, set it equal to a^2; if < 0 equal to $-a^2$; and if $= 0$ forget it. In any event you will arrive after the change of variables $u = x + \frac{b}{2}$ at one of the three integrals

$$\int \frac{1}{u^2+a^2}\,du, \quad \int \frac{1}{u^2-a^2}\,du, \quad \int \frac{1}{u^2}\,du$$

$\int \sqrt{x^2\pm a^2}\,dx = \frac{1}{2}\left(x\sqrt{x^2\pm a^2} \pm a^2\ln\left|x+\sqrt{x^2\pm a^2}\right|\right) + C$

$\int x^n e^{cx}\,dx = x^n\frac{e^{cx}}{c} - \frac{n}{c}\int x^{n-1}e^{cx}\,dx$ etc. This is to be used repetaedly until you arrive at the case $n = 0$, which you can do easily.

17 Hyperbolic Functions

Definition: The most frequently used hyperbolic functions are defined by,

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

where $\sinh x$ and $\tanh x$ are odd like $\sin x$ and $\tan x$, while $\cosh x$ is even like $\cos x$ (see Fig. 4 for plots of these functions). The definitions of other hyperbolic functions are identical to the corresponding trig functions.

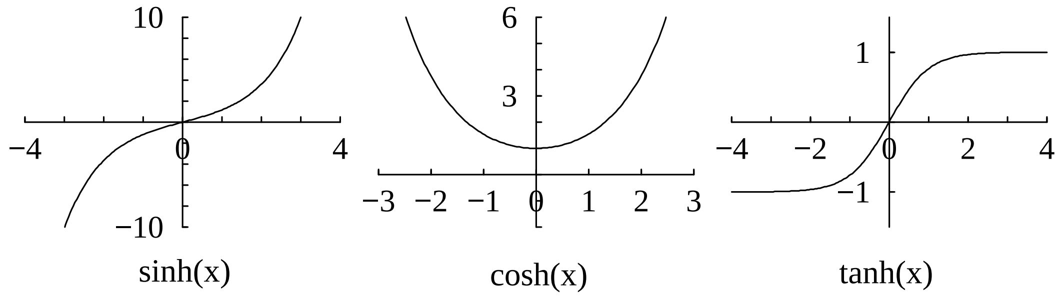

Figure 4: Graphs of $\sinh x$, $\cosh x$, and $\tanh x$.

We can easily derive all the corresponding identities for hyperbolic functions by using the definition. However, it helps us memorize these identities if we know the relationship between $\sinh x$, $\cosh x$ and $\sin x$, $\cos x$ in complex analysis.

Combining $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$, we obtain $\qquad \sinh(x) = -i\sin(ix)$.

Combining $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$, we obtain $\qquad \cosh(x) = \cos(ix)$.

where the constant $i = \sqrt{-1}$ and $i^2 = -1$. Now, we can write down these identities:

$\cos^2 x + \sin^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
$1 + \tan^2 x = \sec^2 x$	$1 - \tanh^2 x = \text{sech}^2 x$
$\cot^2 x + 1 = \csc^2 x$	$\coth^2 x - 1 = \text{csch}^2 x$
$\cos^2 x = (1 + \cos 2x)/2$	$\cosh^2 x = (1 + \cosh 2x)/2$
$\sin^2 x = 1 - \cos^2 x = (1 - \cos 2x)/2$	$\sinh^2 x = \cosh^2 x - 1 = (-1 + \cosh 2x)/2$
$\sin 2x = 2\sin x \cos x$	$\sinh 2x = 2\sinh x \cosh x$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

The Advantage of Hyperbolic Functions

The advantage of hyperbolic functions is that their inverse functions can be expressed in terms of elementary functions. We here show how to solve for the inverse hyperbolic functions.

By definition, $u = \sinh x = \dfrac{e^x - e^{-x}}{2}$.

Multiplying both sides by $2e^x$, we obtain

$2ue^x = (e^x)^2 - 1$ which simplifies to $(e^x)^2 - 2u(e^x) - 1 = 0$.

This is a quadratic in e^x, Thus,

$e^x = [2u \pm \sqrt{(2u)^2 + 4}]/2 = u \pm \sqrt{u^2 + 1}$.

The negative sign in $\pm$ does not make sense since $e^x > 0$ for all x.
Thus, $x = \ln e^x = \ln |u + \sqrt{u^2 + 1}|$ which implies

$x = \sinh^{-1} u = \ln |u + \sqrt{u^2 + 1}|$.

Similarly, by definition, $u = \cosh x = \dfrac{e^x + e^{-x}}{2}$.

We can solve this equation for the two branches of the inverse:

$x = \cosh^{-1} u = \ln |u \pm \sqrt{u^2 - 1}| = \pm \ln |u + \sqrt{u^2 - 1}|$.

Similarly, for

$u = \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

We can solve the equation for the inverse:

$x = \tanh^{-1} u = \dfrac{1}{2} \ln |\dfrac{1+u}{1-u}|$.

The Advantage of Trigonometric Functions

Relations between different trigonometric functions are very important since when we differentiate or integrate one trig function we obtain another trig function. When we use trig substitution, it is often a necessity for us to know the definition of other trig functions that is related to the one we use in the substitution.

Example: Calculate the derivative of $y = \sin^{-1} x$.

Solution: $y = \sin^{-1} x$ means $\sin y = x$. Differentiate both sides of $\sin y = x$, we obtain

$$\cos y y' = 1 \;\Rightarrow\; y' = \frac{1}{\cos y}.$$

In order to express y' in terms of x, we need to express $\cos y$ in terms of x. Given that $\sin y = x$, we can obtain $\cos y = \sqrt{\cos^2 y} = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$ by using trig identities. However, it is easier to construct a right triangle with an angle y. The opposite side must be x while the hypotenuse must be 1, thus the adjacent side is $\sqrt{1 - x^2}$. Therefore,

$$\cos y = \frac{\textit{adjacent side}}{\textit{hypotenuse}} = \frac{\sqrt{1 - x^2}}{1} = \sqrt{1 - x^2}.$$

Example: Calculate the integral $\int \sqrt{1 + x^2} dx$.

Solution: This integral requires standard trig substitution $x = \tan u$ or $x = \sinh u$. Let's use $x = \sinh u$. Recall that $1 + \sinh^2 u = \cosh^2 u$ and that $dx = d \sinh u = \cosh u du$,

$$\int \sqrt{1 + x^2} dx = \int \sqrt{1 + \sinh^2 u} d \sinh u = \int \cosh^2 u du$$

$$= \frac{1}{2} \int [1 + \cosh(2u)] du = \frac{1}{2}[u + \frac{1}{2} \sinh(2u)] + C = \frac{1}{2}[u + \sinh u \cosh u] + C.$$

where hyperbolic identities $\cosh^2 u = [1 + \cosh(2u)]/2$ and $\sinh(2u) = 2 \sinh u \cosh u$ were used. However, we need to express the solution in terms of x. Since $x = \sinh u$ was the substitution, we know right away $u = \sinh^{-1} x = \ln |x + \sqrt{1 + x^2}|$ and $\sinh u = x$, but how to express $\cosh u$ in terms of x? We can solve it using hyperbolic identities. $\cosh u = \sqrt{\cosh^2 u} = \sqrt{1 + \sinh^2 u} = \sqrt{1 + x^2}$. Thus,

$$\frac{1}{2}[u + \sinh u \cosh u] = \frac{1}{2}[\ln |x + \sqrt{1 + x^2}| + x\sqrt{1 + x^2}].$$

However, the following table shows how can we exploit the similarity between trig and hyperbolic functions to find out the relationship between different hyperbolic functions.

Relations between different trig and between different hyperbolic functions

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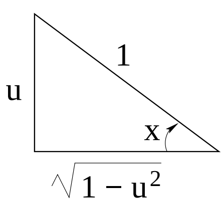
Given one trig func, $\sin x = u = \dfrac{u}{1}$

Draw a right triangle
with an angle x, opp = u, hyp = 1:



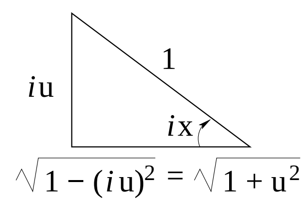
Thus, other trig functions are

$$\cos x = \frac{\text{adj}}{\text{hyp}} = \sqrt{1-u^2}$$

$$\tan x = \frac{\text{opp}}{\text{adj}} = \frac{u}{\sqrt{1-u^2}}$$

Given $\sinh x = -i\sin(ix) = u$, $\sin(ix) = i\text{u} = \dfrac{i\text{u}}{1}$

Draw the corresponding right triangle
with an angle ix, opp = iu, hyp = 1:



By formal analogy,

$$\cosh x = \cos(ix) = \frac{\text{adj}}{\text{hyp}} = \sqrt{1+u^2}$$

$$\tanh x = -i\tan(ix) = -i\frac{\text{opp}}{\text{adj}} = \frac{u}{\sqrt{1+u^2}}$$

==

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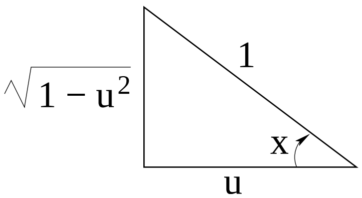
Given that, $\cos x = u = \dfrac{u}{1}$

Draw a right triangle
with an angle x, adj = u, hyp = 1:



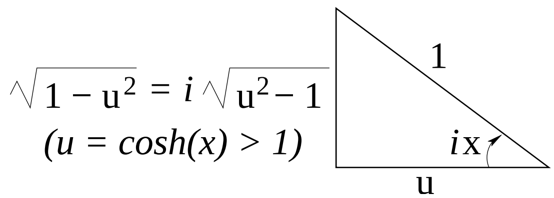
Thus, other trig functions are

$$\sin x = \frac{\text{opp}}{\text{hyp}} = \sqrt{1-u^2}$$

$$\tan x = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{1-u^2}}{u}$$

Given $\cosh x = \cos(ix) = u = \dfrac{u}{1}$

Draw the corresponding right triangle
with an angle ix, adj = iu, hyp = 1:



By formal analogy,

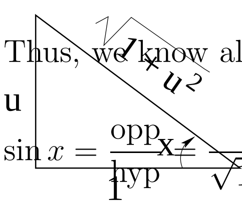
$$\sinh x = -i\sin(ix) = -i\frac{\text{opp}}{\text{hyp}} = \sqrt{u^2-1}$$

$$\tanh x = -i\tan(ix) = -i\frac{\text{opp}}{\text{adj}} = \frac{\sqrt{u^2-1}}{u}$$

==

Given that, $\tan x = u = \dfrac{u}{1}$

Draw a right triangle
with an angle x, opp = u, adj = 1:



Thus, we know all other trig functions

$$\sin x = \frac{\text{opp}}{\text{hyp}} = \frac{u}{\sqrt{1+u^2}}$$

$$\cos x = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{1+u^2}}$$

Given $\tanh x = (-i)\tan(ix) = u$, $\tan(ix) = i\text{u} = \frac{i\text{u}}{1}$

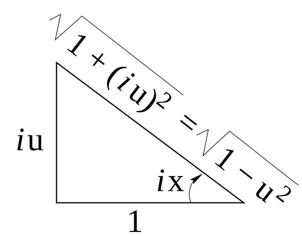
Draw the corresponding right triangle
with an angle ix, opp = iu, adj = 1:

By formal analogy,

$$\sinh x = -i\sin(ix) = -i\frac{\text{opp}}{\text{hyp}} = \frac{u}{\sqrt{1-u^2}}$$

$$\cosh x = \cos(ix) = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{1-u^2}}$$

==


$\sqrt{1+(iu)^2} = \sqrt{1-u^2}$
iu
ix
1